Implementation of a numerical scheme based on the Dual Time Stepping in COSMO LM: idealized test cases

SUMMARY This paper is part of research on the development and testing of a modified version of COSMO LM, that is aimed to assess the feasibility of a time integration technique called Dual Time Stepping (DTS). In a previous paper it was described the implementation of the proposed time integration core with an application for a steady mountain ideal flow. In the present work the DTS scheme has been applied to an unsteady idealized test case, the nonhydrostatic inertia-gravity wave, that involves the evolution of a potential temperature perturbation in a channel. This study on idealized cases will be followed, in future research, by an assessment of the proposed numerical scheme on a realistic test case for Meteorology and Climatology.
INTRODUCTION AND MOTIVATIONS

As stated in a previous research paper [1], no evidence of a previous attempt to use Dual Time Stepping (DTS) in Numerical Weather Prediction (WPN) has been found in literature by the author. In order to reduce the complexity of the re-coding effort, the original framework of COSMO [2] has been used with a limited amount of modifications. The aim of the present work is to assess the use of the DTS scheme on an idealized test case, extensively studied in the reference literature [3, 4], by comparing the results obtained with the original version of COSMO. Therefore the non-hydrostatic inertia-gravity wave test is particularly interesting since it allows to compare the evolution of a gravity wave without any disturbance introduced by the parametrizations and/or the orography. The original version of COSMO adopts the time-splitting scheme of Wicker and Skamarock [6], where the slow processes are integrated with a second-order three-stage RK scheme and the fast ones with a forward-backward scheme in the horizontal direction and an implicit Crank-Nicholson scheme in vertical. In the DTS version of COSMO all (fast and slow) processes are treated using the same scheme.

INERTIA-GRAVITY WAVES

The first test case that we use was proposed by Skamarock et al. [3]. The nonhydrostatic inertia-gravity wave involves the evolution of a potential temperature perturbation in a channel with a background flow. A lot of meteorological models, like COSMO, discretise and integrate the advection and the fast modes (sound, gravity waves) differently. The introduction of a background flow serves as a test for a good coupling of these processes. The initial conditions we use are identical to those of Giraldo [4]. The initial state of the atmosphere is taken to have a constant mean flow in a uniformly stratified atmosphere with a constant Brunt- Väisälä frequency. Skamarock and Klemp gave an analytic solution for this test but, unfortunately, it is only valid for the Boussinesq equations. Therefore we compare only the maximum and minimum values of potential temperature perturbation and vertical velocity perturbation with those reported by Giraldo. Different settings of COSMO were examined.

The second test case that we use is based on the work of Baldauf [5], where a slightly modified version of the idealised test setup used by Skamarock and Klemp is proposed: the modification allows to derive an exact analytical solution for the compressible, non-hydrostatic Euler equations, for the quasi linear 2-dimensional expansion of sound and gravity waves in a channel induced by a weak warm bubble.

We introduce here some general remarks about the numerical simulation. Every physical parameterisation in the numerical model must be switched off, in particular (turbulent) diffusion and boundary layer treatment. Furthermore, free slip boundary conditions must be prescribed at the top and the bottom of the domain. Damping layers at the lateral boundaries and Rayleigh damping must be switched off, too. To be in a strongly dominated linear regime, the initial temperature perturbation must be relatively small (0.01 K). Consequently, only small pressure perturbations are produced which must be compared with the total pressure $O(10^5 \text{ Pa})$. Thus double precision for all floating point operations is recommended.

INERTIA-GRAVITY WAVES IN THE ORIGINAL COSMO

In the first test case the initial state of the atmosphere is taken to have a constant horizontal mean flow of $u = 20 \text{ m/s}$ in a uniformly strat-
ified atmosphere with a Brunt–Väisälä frequency of $N = 0.01/s$. The domain is defined as $(x, z) \in [0, 300000] m \times [0, 10000] m$ with $t \in [0, 3000] s$. Figure 1 shows the iso-contours of potential temperature perturbation at the beginning of the simulation, where a hot bubble ($\Delta \Theta = 0.01^\circ C$) is placed at $x = 100000$ m, following the same perturbation law used by Giraldo. No-flux boundary conditions are used along the bottom and top boundaries, that are modeled as a rigid-lid-free-slip. At the lateral boundaries inflow/outflow free-slip conditions are imposed. The results are considered after 3000 s of simulation, when the initial disturbance has been propagated. A first study has been carried out on the original version of COSMO in order to assess the grid convergence and analyze the sensitivity of the solution to the numerical parameters used in the time integration scheme. More in detail, it has been assessed the impact of the weighting applied to the numerical time derivative in the fast waves implicit scheme. Further studies have been carried out using different advection orders, but no significant changes have been noticed and the results have been not reported. A complete summary of the results can be found in Tables 1 and 2. The solution corresponding to the case labeled as OR$^{\beta}_{9}$ has been reported in the following figures.

The iso-contours of potential temperature perturbation ($\Delta \Theta$) after 3000 s of simulation are reported in Figure 2, showing the symmetry of the waves about the position $x = 160000$ m.

The symmetry can be either observed in Figure 3 where the iso-contours of the vertical velocity ($w$) have been reported. We use the same contouring interval used in Giraldo and our results look quite similar.

Figure 4 shows the iso-contours of horizontal velocity perturbation ($\delta u$) after 3000 s of simulation, using the original routines of COSMO: it is possible to notice the simmetrical behaviour of the inertia-gravity waves.

The distribution of the potential temperature perturbation ($\Delta \Theta$) has been reported at an altitude of $z=5000$ m in Figure 5. It is interesting to notice the particular behaviour of the potential temperature perturbation around the position $x = 160000$ m, where a certain time resolution is required to obtain this two-small-peaks structure. This aspect will be analyzed in the following sections where the dependency of the solution on the time resolution will be assessed.
GRID CONVERGENCE STUDY

In this section we analyze the impact of the spatial resolution (\(\Delta x\) and \(\Delta z\)) and time resolution (dt) on the results obtained with the original routines of COSMO. Several simulations were carried out to assess the grid convergence on the maximum potential temperature perturbation (\(\max \Delta \Theta\)), the minimum potential temperature perturbation (\(\min \Delta \Theta\)), the maximum vertical velocity (max w) and the minimum vertical velocity (min w) after 3000s of simulation, as reported in Table 1. The reference results are those reported by Giraldo, with a horizontal and vertical resolution of 0.25 km and a time step of 12 s.

The cases labeled as OR2, OR5, OR6, OR7 and OR8 show the dependency of the solution on the time step, with dt ranging from 3 s to 50 s, with a constant horizontal resolution of \(\Delta x=1\) km and \(\Delta z=0.5\) km. The comparison among the cases labeled as OR1, OR2, OR3 and OR4 show the dependency of the solution on the vertical resolution with \(\Delta z\) ranging from 1 km to 0.125 km and a constant \(\Delta x=1\) km, dt=12 s. Furthermore the dependency on the horizontal resolution can be assessed by comparing the cases labeled as OR1, OR9, OR10 and OR3, OR13. The potential temperature perturbation (\(\Delta \Theta\)) converges toward the results obtained by Giraldo as the resolution increases, while the vertical velocity perturbation remains different even for the finest resolution. The results show that the dependency of the solution on the horizontal and vertical resolution becomes negligible with a grid spacing smaller than 0.25 km, while the dependency on the time step seems more tricky. Two effects have to be considered. At first, the fast (sound and gravity) waves in COSMO are integrated with a smaller and roughly constant time step, evaluated indepen-
A Dual Time Stepping (DTS) integration for COSMO LM

...tently on the input (longer) time step. Moreover the fast waves time derivative depends on the weighting parameters adopted in the implicit formulation.

Finally it has been noticed that the grid convergence to the solution obtained by Giraldo is almost of second order for the potential temperature perturbation, while the vertical velocity perturbation does not show the same behaviour. In order to produce a more consistent comparison of the results with the ones obtained by Giraldo, the case labeled as OR13 has been chosen to perform a further sensitivity study on the values used for time-weighting in the treatment of acoustic (sound) waves and gravity waves.

**SENSITIVITY ON THE TIME-WEIGHTING PARAMETERS**

In this section we analyze the impact of the time weighting parameters of COSMO on the results obtained with the original routines of COSMO. We recall here that $\beta_{sw}$ is the value of the $\beta$-parameter used for time-weighting of the future values in the vertically implicit treatment of acoustic (sound) waves. Indeed $\beta_{sw}=0$ gives a time-centred average with no damping, $\beta_{sw}=1$ results in a fully implicit vertical scheme with strong damping of acoustic and gravity wave modes. The parameter $\beta_{gw}$ is the same as $\beta_{sw}$, but used for gravity waves. Both these parameters act on the $w$-velocity equation. On the other hand we have the parameter $\beta_{2sw}$, same as $\beta_{sw}$ but used in the $p^*$, $T^*$ dynamics for sound waves, and the parameter $\beta_{2gw}$, same as $\beta_{gw}$ but used in the $p^*$, $T^*$ dynamics for gravity waves. Since slight positive off-centering is recommended to damp disturbances, COSMO uses a default value of 0.4 for all these parameters. We noticed that the values of $(\beta_{sw}/\beta_{2sw}/\beta_{gw}/\beta_{2gw})$ highly impact the symmetry and amplitude of the waves in the solution obtained after 3000s of simulation. Therefore we performed the sensitivity study reported in Table 2. Figure 6 shows the iso-contours of horizontal velocity perturbation ($\delta u$) after 3000s of simulation for the case labeled as OR13. It’s possible to notice that using $(\beta_{sw}/\beta_{2sw}/\beta_{gw}/\beta_{2gw})=0/0/0/0)$ results in a presence of high-frequency noise in the horizontal velocity distribution.

![Figure 6](image1.png)

*Source: Original COSMO*

![Figure 7](image2.png)

*Source: Original COSMO*

This effect is highlighted in the distribution of the $w$-velocity, reported in Figure 7. At
least a non-zero value of one element among \((\beta_{sw}/\beta_{gw})\) is required to damp these oscillations, while better results and symmetrical behaviour can be achieved reducing them with respect to the default settings.

In order to obtain a better agreement with the results by Giraldo, the case labeled as \(\text{OR}_{\beta9}\), where only \(\beta_{2sw}=0.4\), has been chosen in order to compare the original routines of COSMO with the DTS modifications introduced in this paper.

**INERTIA-GRAVITY WAVES IN THE DTS COSMO**

Figure 8 shows the iso-contours of horizontal velocity perturbation \((\delta u)\) after 3000 s of simulation, using the modified DTS routines of COSMO: it is possible to notice that the structure of the u-velocity field corresponds to the one obtained with the original routines, in Figure 4. For a sake of clarity the same contour intervals have been used. Comparing the w-velocity fields, i.e. Figure 9 and Figure 3 and the potential temperature perturbation fields, i.e. Figure 10 and Figure 2, it is possible to notice that the DTS predictions using a physical time step of \(dt=3\) s give results that are consistent with the ones obtained in the case labeled as \(\text{OR}_{\beta9}\) in the original COSMO.

**DUAL TIME RESOLUTION CONVERGENCE STUDY**

The time-splitting formulation implemented in the original COSMO takes advantage of a time accuracy in the resolution of the fast processes given by the use of small time steps. Since we're considering a gravity (fast) waves test case, each physical time step in DTS COSMO has to be small enough to ensure a good resolution of the fast processes involved in the governing equations. In order to assess the minimum requirements for DTS COSMO, time convergence study has been carried out and the results collected in Table 3. For a better understanding of the impact of the dual time resolution in comparison with the reference case \((\text{OR}_{\beta9})\), the distribution of the potential temperature perturbation \((\Delta \Theta)\) has been reported at an altitude of \(z=5000\) m in Figure 11. Comparing the cases labeled as \(\text{DTS1}(dt=12\) s)\), \(\text{DTS2}(dt=6\) s\) and \(\text{DTS3}(dt=3\) s\), it is possible to notice that a time step of \(dt=3\) s is required to obtain a good agreement of the potential temperature perturbation with the reference case. A detail of the potential temperature perturbation at a height of 5000 m among the latitudes of \(x=155000\) m...
and $x=165000$ m has been reported in Figure 16. A dual time step of $dt=12$ s is not enough to capture the two peaks of potential temperature perturbation, while the results get closer to the
ANALYTICAL SOLUTION FOR THE LINEAR GRAVITY WAVES

In the second test case we consider the evolution of a small initial deviation from a stratified atmosphere which is contained in a two-dimensional channel. The stratified atmosphere is isothermal with the absolute temperature $T_0$, which leads to a constant Brunt–Väisälä frequency and a constant sound speed. Under this assumption an analytical solution can be derived for linear gravity waves by means of a Bretherton\[10\] transformation, as shown by Baldauf \[9\]. At the top and the bottom free slip boundary conditions are used. At the lateral boundaries periodic boundary conditions are used. The equations and the boundary conditions are satisfied for a stationary, hydrostatic, horizontally homogeneous, and isothermal background state. The warm bubble is initialised by the following deviations from the background atmosphere:

$$
T'(r,t=0) = e^{\pm \frac{1}{2} \delta z} \cdot T_b(r,t=0) \\
\rho'(r,t=0) = e^{\pm \frac{1}{2} \delta z} \cdot \rho_b(r,t=0) \\
\rho'(r,t=0) = e^{\pm \frac{1}{2} \delta z} \cdot \rho_b(r,t=0)
$$

(1)

with the Bretherton-transformed temperature deviation

$$
T_b(r,t=0) = \Delta T \cdot e^{-\frac{(x-x_c)^2}{a^2}} \cdot \sin \frac{z}{H}
$$

(2)

where $\Delta T$ is the temperature perturbation, $x_c$ is the initial position of the warm bubble, $a$ the radius of the bubble, $H$ the height of the chan-
nel and \( \delta = \frac{2}{RT_0} \) the Bretherton-height parameter. The appropriate density deviation can be computed from the linearised ideal gas law. A Gaussian shape of the bubble in horizontal direction has been implemented in COSMO, as shown in [5].

**ANALYTICAL SOLUTION VS ORIGINAL COSMO SOLUTION**

In the following we report the comparison between the potential temperature perturbation \((\Delta \Theta)\) and the w-velocity perturbation \((\Delta w)\), after 1800s, using four different grid resolutions (see Table 4) in the original COSMO. The test case was carried out with the following parameters for the warm bubble: \( \Delta T = 0.01 \) K, the radius \( d = 5000 \) m, and the initial position \( x_c = 100 \) km. The domain is defined as \((x, z) \in [0, 300000] m \times [0, 10000] m\) with \( t \in [0, 1800] s\).

For the background atmosphere, we take \( T_0 = 250 \) K and \( p_{sl} = 10^5 \) Pa at sea level. In addition, we use the following constants for the dry air \( R = 287.05 \) J/kg/K, \( c_p = 1005.0 \) J/kg/K, \( c_v = c_p - R \), and the gravity acceleration as \( g = 9.80665 \) m/s\(^2\). We prescribe a constant background flow of \( u_0 = 20 \) m/s. A constant vertical resolution of the grid of \( \Delta z = 125 \) m has been fixed. The weighting parameters of the time-derivative \((\beta_{sw}/\beta_{gw})\) have been set equal to zero, while \( \beta_{gw} \) equals 0.1 to avoid numerical high-frequency oscillations of the results. A small modification has been introduced in COSMO to reduce the small time step of the fast waves by five times with respect to its standard value, in order to obtain an enhanced time resolution of the gravity waves. Figures 15, 16, 17 and 18 show the convergence of the potential temperature perturbation and the w-velocity perturbation to the analytical solution at a height of 5000 m. As proposed in [5], to quantify the error of a solution \( \Psi \) of COSMO, we use the mean value of the error in a suitable norm:

\[
L^2(\Psi) = \left( \frac{1}{N_x N_z} \sum_{i=1}^{N_x} \sum_{k=1}^{N_z} |\Psi - \Psi_{ref}|^2 \right)^{\frac{1}{2}}
\]

\[
L^\infty(\Psi) = \max |\Psi - \Psi_{ref}|
\]

The grid convergence using the \( L^2\)-norm and \( L^\infty\)-norm of the potential temperature perturbation and the w-velocity perturbation have been reported in Figures 13 and 14 using respectively the finest grid and the analytical solution as reference.
Table 4
Analytic grid convergence in original COSMO: inertia-gravity waves

<table>
<thead>
<tr>
<th>GRID</th>
<th>Δx [m]</th>
<th>dt [s]</th>
<th>$L^\infty$ error on $\Delta \Theta$</th>
<th>$L_2$ error on $\Delta \Theta$</th>
<th>$L^\infty$ error on $\Delta w$</th>
<th>$L_2$ error on $\Delta w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1000</td>
<td>20.0</td>
<td>0.3681·10^{-3}</td>
<td>0.7450·10^{-4}</td>
<td>0.1022·10^{-2}</td>
<td>0.2891·10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>500</td>
<td>10.0</td>
<td>0.1299·10^{-3}</td>
<td>0.3566·10^{-4}</td>
<td>0.0396·10^{-2}</td>
<td>0.1201·10^{-3}</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
<td>5.0</td>
<td>0.1159·10^{-3}</td>
<td>0.2819·10^{-4}</td>
<td>0.0211·10^{-2}</td>
<td>0.0559·10^{-3}</td>
</tr>
<tr>
<td>4</td>
<td>125</td>
<td>2.5</td>
<td>0.1157·10^{-3}</td>
<td>0.2596·10^{-4}</td>
<td>0.0106·10^{-2}</td>
<td>0.0273·10^{-3}</td>
</tr>
</tbody>
</table>
The importance of considering a variation of the weighting parameters with respect to the default values of 0.4 is shown comparing the Figure 18 and Figure 19, obtained for the finest grid.

Figure 19: potential temperature perturbation ($\Delta \Theta$ [K]) and w-velocity perturbation ($\Delta w$) after 1800 s: grid 4 vs Analytic solution. ($\beta_{sw}/\beta_{sw}/\beta_{gw}/\beta_{gw}$) = (0.4/0.4/0.4/0.4) and original small time step.

Source: original COSMO

The importance of considering a reduction of the small time step used in the fast processes with respect to the value computed by COSMO is shown comparing the Figure 18 and Figure 20, obtained for the finest grid.

ANALYTICAL SOLUTION VS DTS COSMO SOLUTION

In the following we report the comparison between the potential temperature perturbation ($\Delta \Theta$) and the w-velocity perturbation ($\Delta w$), after 1800 s, using three different grid resolutions (see Table 5) in the DTS COSMO.

When not explicitly specified, the previous settings used in the original COSMO have been kept. The convergence threshold in DTS COSMO has been set to a decay of three logarithmic levels in the residual of the pressure perturbation. The time step used in DTS COSMO equals to $dt=0.36$ s, the same time step used in the original COSMO to handle the fast processes in the governing equation. Figures 23, 24 and 25 show the convergence of the potential temperature perturbation and the w-velocity perturbation to the analytical solution at a height of 5000 m.

Figure 20: potential temperature perturbation ($\Delta \Theta$ [K]) and w-velocity perturbation ($\Delta w$) after 1800 s: grid 4 vs Analytic solution. ($\beta_{sw}/\beta_{sw}/\beta_{gw}/\beta_{gw}$) = (0/0/0.1/0.1) and original small time step.

Source: original COSMO

The grid convergence using the $L^2$-norm and $L^\infty$-norm of the w-velocity perturbation and the w-velocity perturbation have been reported in Figures 21 and 22 using respec-
Table 5
Analytic grid convergence in DTS COSMO: inertia-gravity waves

<table>
<thead>
<tr>
<th>GRID</th>
<th>∆x [m]</th>
<th>dt [s]</th>
<th>$L_\infty$ error on $\Delta \Theta$</th>
<th>$L_2$ error on $\Delta \Theta$</th>
<th>$L_\infty$ error on $\Delta w$</th>
<th>$L_2$ error on $\Delta w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1000</td>
<td>0.36</td>
<td>$2.161 \cdot 10^{-3}$</td>
<td>$5.499 \cdot 10^{-4}$</td>
<td>$5.031 \cdot 10^{-4}$</td>
<td>$1.608 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>500</td>
<td>0.36</td>
<td>$1.158 \cdot 10^{-3}$</td>
<td>$3.123 \cdot 10^{-4}$</td>
<td>$2.544 \cdot 10^{-4}$</td>
<td>$7.434 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
<td>0.36</td>
<td>$1.160 \cdot 10^{-3}$</td>
<td>$2.718 \cdot 10^{-4}$</td>
<td>$1.305 \cdot 10^{-4}$</td>
<td>$3.407 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

Analytically the finest grid and the analytical solution as reference. Comparing Table 4 and Table 5, it's possible to notice that using the same time resolution in the fast and slow processes lead to
a better agreement with analytic results, even when using the coarsest grid. To conclude this paper we report in Figure 26 the solution obtained on the coarsest grid using a decay of two logarithmic levels in the residual of the pressure perturbation. Comparing the results with those obtained in Figure 23, it is possible to notice the importance of ensuring a certain accuracy in the prediction of the pressure perturbation to obtain a better comparison with the analytic results. The author recommends at most three logarithmic decay in the pressure perturbation residual since additional studies, not reported in this paper, have shown no significant improvement in the quality of the results when further increasing this threshold. The pressure perturbation residual has been adopted to control the DTS convergence since it’s the slowest converging one due to preconditioning.

CONCLUSIONS

In this paper the numerical performances of the DTS have been assessed in comparison with the operational COSMO model Runge Kutta scheme on an ideal test case. The inertia-gravity waves case shows that the DTS integration is able to give comparable results to the existing COSMO, both in terms of accuracy and mesh convergence, leading the author to further investigate the advantages and disadvantages of the proposed methodology with a real test case in Meteorology and Climatology.

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[1] Petrone, G., Theoretical study and software design for the implementation of a numerical scheme based on the Dual Time Stepping in COSMO LM, CMCC Research Papers Issue RP0135


